Infinite connected sums, K-area and positive scalar curvature

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8th February 2008

Abstract

Recently, Whyte [W] used the index theory of Dirac operators and the Block-Weiberger uniformly finite homology [BW] to show that certain infinite connected sums do not carry a metric with nonnegative scalar curvature in their bounded geometry class. His proof uses a generalization of the $\hat{A}$-class to obstruct such metrics. In this note we prove a variant of Whyte's result where infinite K-area in the sense of Gromov [G1] is used to obstruct metrics with positive scalar curvature.

1 Introduction

We will consider the category $\mathcal{BG}_n$ of manifolds with bounded geometry, i.e. objects in $\mathcal{BG}_n$ are complete n-dimensional Riemannian manifolds whose curvature tensor and covariant derivatives of all orders are uniformly bounded and whose injectivity radius is positive. The morphisms of $\mathcal{BG}_n$ are diffeomorphisms with bounded distortion, so that the natural action on metrics preserves the bounded geometry structure. By passing to the quotient we obtain the so-called *bounded geometry classes* of metrics. Whenever we refer to a manifold, it should be understood that it is equipped with a metric varying within a fixed bounded geometry class. Also, all manifolds in the paper will be spin, unless otherwise stated.

We observe that in [BW] a real homology theory H_0^{uf}, named uniformly finite homology in degree zero, has been defined which is preserved under the above morphisms and therefore is a bounded geometry invariant. In particular, a subset $S \subset X$ defines a class $[S] \in H_0^{uf}$ if it is *locally uniformly finite* in the sense that

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for each $r > 0$ there exists $C_r > 0$ such that the amount of points of S inside any metric ball of radius r is bounded from above by C_r (for more on the functor H_0^{uf} and its relation to bounded de Rham cohomology, see Section 4).

Now take $Y \in \mathcal{BG}$, Z a closed manifold and $S \subset Y$ as above. Let $Y \sharp_S Z$ be the manifold obtained by connected summing to Y a copy of Z along a small neighborhood of each element of S.

Remark 1.1. *Notice that $\mathcal{BG}$ is stable under such infinite connected sums, so that $Y \sharp_S Z \in \mathcal{BG}$. Also, H_0^{uf} is in fact a coarse invariant so we have a natural identification $H_0^{uf}(Y \sharp_S Z) \cong H_0^{uf}(Y)$.*

Recently, Whyte [W] used the index theory of Dirac operators to show that if Y carries a metric of nonnegative scalar curvature (in a given bounded geometry class), $\hat{A}(Z) \neq 0$ and $[S] \neq 0$ then $Y \sharp_S Z$ does not carry a metric of nonnegative scalar curvature (in the corresponding bounded geometry class).

Example 1.1. *Let $S = \mathbb{Z}^n \subset \mathbb{R}^n$ be the standard lattice in flat Euclidean space. Then $[\mathbb{Z}^n] \neq 0$ and hence $\mathbb{R}^{4l} \sharp_{\mathbb{Z}^{4l}} Z^{4l}$ does not carry a metric with nonnegative scalar curvature if $\hat{A}(Z) \neq 0$ (for example we can take Z equal to the product of l Kummer surfaces). More generally, we can take $Y \to Y_0$ to be an infinite covering with Y_0 closed and carrying a metric of nonnegative scalar curvature (Y is equipped with the covering metric). Then Whyte's result applies to $Y \sharp_S Z$ if $S \subset Y$ is an orbit under deck transformations and $\pi_1(Y_0)$ is amenable (for more on Whyte's result, see Remark 2.1).*

The purpose of this note is to prove a variant of Whyte's result which uses the assumption of infinite K-area in the sense of Gromov [G1] to obstruct metrics with positive (but not necessarily uniformly positive) scalar curvature.

Theorem 1.1. *Let $Y \in \mathcal{BG}_{2k}$ admit a metric of positive scalar curvature in its bounded geometry class and let $S \subset Y$ with $[S] \neq 0$ in $H_0^{uf}(Y)$. Then if $K_{\mathrm{area}}(Z) = +\infty$ then $Y \sharp_S Z$ does not carry a metric of positive scalar curvature in its bounded geometry class.*

Our theorem follows immediately from the propositions below.

Proposition 1.1. *If $X \in \mathcal{BG}_{2k}$ carries a metric of positive scalar curvature then $K_{\mathrm{area}}(X) < +\infty$.*

Proposition 1.2. *Let $Y \in \mathcal{BG}_{2k}$ with $K_{\mathrm{area}}(Y) < +\infty$. Then $K_{\mathrm{area}}(Y \sharp_S Z) = +\infty$ if $K_{\mathrm{area}}(Z) = +\infty$ and $[S] \neq 0$.*

Remark 1.2. *Proposition 1.1 has an independent interest as it shows that infinite K-area is an obstruction to the existence of metrics of positive scalar curvature in the bounded geometry framework. This applies notably to certain large Riemannian manifolds (see Remark 3.2).*

Remark 1.3. *Notice that the assumption $\hat{A}(Z) \neq 0$ only makes sense if $\dim Z$ (and hence $\dim Y$) is a multiple of four. On the other hand, our result applies to certain manifolds in every even dimension and moreover the attached manifold Z can be chosen to be more familiar. It applies for example to $(V^{2k-2} \times P^2)\sharp_S \mathbb{T}^{2k}$, where V is any flat manifold, $P^2 \subset \mathbb{R}^3$ is the standard paraboloid of revolution, $\mathbb{T}^{2k}$ is a torus (which has infinite K-area) and S is chosen suitably (see Remark 4.1). More generally, we could replace $\mathbb{T}^{2k}$ by any finitely enlargeable spin manifold [G1]. Also, note that as remarked in [W], the class $[S]$ lies in a non-Hausdorff homology group and hence standard obstructions based on C^*-algebra techniques do not seem to work here.*

Our proof of Theorem 1.1 follows Whyte's approach with suitable modifications to account for the fact that we will be dealing with almost flat complex bundles over X. The presentation emphasizes the use of the index theory of generalized Atiyah-Patodi-Singer (APS) type boundary conditions (see Section 2). Combined with a twisted version of an integral identity derived in [HMZ], this allows us to establish a vanishing result (Proposition 2.2) for twisted APS harmonic spinors. The concept of K-area is reviewed in Section 3 and in Section 5 we give the proofs of Propositions 1.1 and 1.2 above.

2 APS index theory and a vanishing result for harmonic spinors

If W is an oriented n-dimensional spin manifold with a fixed spin structure [LM] and Riemannian metric then there exists over W a canonical hermitian vector bundle S_W, the spinor bundle, which comes equipped with a Clifford product $\gamma : \Gamma(TW) \to \Gamma(\mathrm{End}(S_W))$ and a compatible connection $\nabla : \Gamma(S_W) \to \Gamma(T^*W \otimes S_W)$. Using these structures we can define the corresponding Dirac operator $\partial\!\!\!/ : \Gamma(S_W) \to \Gamma(S_W)$ acting on spinors,

$$\partial\!\!\!/ = \sum_{i=1}^{n} \gamma(\mathrm{e}_i)\nabla_{\mathrm{e}_i}, \tag{2.1}$$

where $\{\mathrm{e}_i\}$ is a local orthonormal basis tangent to W. More generally, we can fix a hermitian vector bundle $\mathcal{E}$ with compatible connection ∇ and consider the twisted

Dirac operator $\partial\!\!\!/_{\mathcal{E}} : \Gamma(S_W \otimes \mathcal{E}) \to \Gamma(S_W \otimes \mathcal{E})$ acting on (twisted) spinors. The Weitzenböck decomposition for the corresponding Dirac Laplacian is

$$\partial\!\!\!/_{\mathcal{E}}^2 = \nabla^*\nabla + \frac{1}{4}\kappa + \mathcal{R}^{[\mathcal{E}]}, \tag{2.2}$$

where $\nabla^*\nabla$ is the Bochner Laplacian of $S_W \otimes \mathcal{E}$, κ is the scalar curvature of W and for $\psi \otimes \eta \in \Gamma(S_W \otimes \mathcal{E})$,

$$\mathcal{R}^{[\mathcal{E}]}(\psi \otimes \eta) = \frac{1}{2}\sum_{ij} \gamma(\mathrm{e}_i)\gamma(\mathrm{e}_j)\psi \otimes R^{\mathcal{E}}_{\mathrm{e}_i,\mathrm{e}_j}\eta, \tag{2.3}$$

with $R^{\mathcal{E}}$ being the curvature tensor of ∇. If W is closed, $\partial\!\!\!/_{\mathcal{E}}$ is a self-adjoint elliptic operator and $\ker \partial\!\!\!/_{\mathcal{E}}$, the space of harmonic spinors, has finite dimension.

If $n = 2k$ one has a decomposition

$$S_W \otimes \mathcal{E} = \left(S_W^+ \otimes \mathcal{E}\right) \oplus \left(S_W^- \otimes \mathcal{E}\right) \tag{2.4}$$

into positive and negative spinors induced by the chirality operator Υ and $\partial\!\!\!/_{\mathcal{E}}$ interchanges the factors, so we can decompose, according to (2.4),

$$\partial\!\!\!/_{\mathcal{E}} = \begin{pmatrix} 0 & \partial\!\!\!/_{\mathcal{E}}^- \\ \partial\!\!\!/_{\mathcal{E}}^+ & 0 \end{pmatrix}, \tag{2.5}$$

where

$$\partial\!\!\!/_{\mathcal{E}}^{\pm} = \partial\!\!\!/_{\mathcal{E}}|_{\Gamma(S_W^{\pm} \otimes \mathcal{E})} : \Gamma(S_W^{\pm} \otimes \mathcal{E}) \to \Gamma(S_W^{\mp} \otimes \mathcal{E}). \tag{2.6}$$

This gives $\ker \partial\!\!\!/_{\mathcal{E}} = \ker \partial\!\!\!/_{\mathcal{E}}^+ \oplus \ker \partial\!\!\!/_{\mathcal{E}}^-$ and moreover $\partial\!\!\!/_{\mathcal{E}}^+$ and $\partial\!\!\!/_{\mathcal{E}}^-$ are adjoints to each other, so we get a well-defined index

$$\mathrm{ind}\, \partial\!\!\!/_{\mathcal{E}}^+ = \dim\ker \partial\!\!\!/_{\mathcal{E}}^+ - \dim\ker \partial\!\!\!/_{\mathcal{E}}^-. \tag{2.7}$$

The Atiyah-Singer index formula computes this integer as

$$\mathrm{ind}\, \partial\!\!\!/_{\mathcal{E}}^+ = \int_W [\hat{A}(TW) \wedge \mathrm{ch}(\mathcal{E})]_{2k}, \tag{2.8}$$

where $\mathrm{ch}(\mathcal{E}) \in H^{2*}(W;\mathbb{Q})$ is the Chern character of $\mathcal{E}$, $\hat{A}(TW) \in H^{4*}(W;\mathbb{Q})$ is the $\hat{A}$-class of TW and the notation $[\ \ ]_{2k}$ means that integration picks the element of degree $2k$ in the wedge product. Specializing to the case where $k = 2l$ and $\mathcal{E} = W \times \mathbb{C}$ is the trivial line bundle (equipped with the standard flat connection) we get

$$\mathrm{ind}\, \partial\!\!\!/^+ = \hat{A}(W), \tag{2.9}$$

where

$$\hat{A}(W) = \int_W [\hat{A}(TW)]_{4l} \tag{2.10}$$

is the $\hat{A}$-genus of W. Notice that in this case (2.2) reduces to

$$\partial\!\!\!/^2 = \nabla^*\nabla + \frac{\kappa}{4}. \tag{2.11}$$

Remark 2.1. *Recall that the famous Lichnerowicz's argument [L] is based on the fact that, since $\nabla^*\nabla$ is nonnegative, the positivity of κ in (2.11) implies that $\partial\!\!\!/$ is positive and hence invertible, which gives $\hat{A}(W) = 0$ by (2.7) and (2.9). So, $\hat{A}(W) \neq 0$ is a topological obstruction to the existence of metrics with $\kappa > 0$. Thus, the point of Whyte's theorem is that if an obstructing Z (i.e. with $\hat{A}(Z) \neq 0$) is 'glued' to Y along a non null class $[S] \in H_0^{uf}(Y)$ then an obstruction to metrics with $\kappa \geq 0$ on $Y\sharp_S Z$ arises, even if Y originally carries such a metric. And our main result of course says that instead of nonzero $\hat{A}$-genus we can use infinite K-area as a 'glued' obstruction to metrics of positive scalar curvature.*

We now consider the index theory for manifolds with boundary (see for example [APS, BoW, G, S] and the references therein). Assume from now on that W is a compact spin manifold with dimension $n = 2k$ and *nonempty* smooth boundary $\Sigma \subset W$, and $\mathcal{E}$ is a hermitian bundle over W with a compatible connection. The point is that the computation of the righthand side of (2.8) in this more general setting also leads to an interesting index problem. To see this, introduce geodesic coordinates $(x, u) \in \Sigma \times [0, \delta) \to \mathcal{U}$ in a collar neighborhood $\mathcal{U}$ of Σ and set $\Sigma_u = \{(x, u); x \in \Sigma\}$ so that $\Sigma_0 = \Sigma$. Then, restricted to $\mathcal{U}$,

$$\partial\!\!\!/ = \gamma\left(\frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial u} + D - \frac{1}{2}H\right), \tag{2.12}$$

where H is the mean curvature of the embeddings $\Sigma_u \subset \mathcal{U}$ and D is a self-adjoint linear operator, the tangential Dirac operator, defined as follows. For each u, $S_W|_{\Sigma_u}$ comes equipped with the Clifford product $\gamma^u = -\gamma(\partial/\partial u)\gamma$, so if we consider the induced connection

$$\nabla^u = \nabla - \frac{1}{2}\gamma^u(A) = \nabla + \frac{1}{2}\gamma\left(\frac{\partial}{\partial u}\right)\gamma(A),$$

where A is the shape operator of the embedding $\Sigma_u \subset \mathcal{U}$, then

$$D = \sum_{i=1}^{2k-1} \gamma^u(\mathrm{e}_i)\nabla^u_{\mathrm{e}_i},$$

where $\{\mathrm{e}_i\}$ is an orthonormal basis tangent to Σ_u. After twisting with $\mathcal{E}$, $D_{\mathcal{E}}$ is a first order self-adjoint elliptic operator acting on $S_\Sigma \otimes \mathcal{E}$ and commuting with Υ, so we can decompose $S_W \otimes \mathcal{E}|_\Sigma =: \mathbb{S}_{\mathcal{E}} = \mathbb{S}_{\mathcal{E}}^+ \oplus \mathbb{S}_{\mathcal{E}}^-$ and accordingly,

$$D_{\mathcal{E}} = \begin{pmatrix} D_{\mathcal{E}}^+ & 0 \\ 0 & D_{\mathcal{E}}^- \end{pmatrix}, \tag{2.13}$$

with $D_{\mathcal{E}}^\pm$ selfadjoint. Under the natural identification $\mathbb{S}_{\mathcal{E}}^+ = \mathbb{S}_{\mathcal{E}}^-$ one has $D_{\mathcal{E}}^+ = -D_{\mathcal{E}}^-$ and hence $\mathrm{Spec}(D_{\mathcal{E}})$ is symmetric with respect to $0 \in \mathbb{R}$, but of course this does not need happen to the factors $D_{\mathcal{E}}^\pm$. Thus, for $\mathrm{Re}\, z \gg 0$ we define the eta function

$$\eta_{\mathcal{E}}^+(z) = \sum_{0 \neq \lambda \in \mathrm{Spec}(D_{\mathcal{E}}^+)} \mathrm{sign}\, \lambda |\lambda|^{-z} \dim E_\lambda(D_{\mathcal{E}}^+),$$

where $E_\lambda(D_{\mathcal{E}}^+)$ is the eigenspace of $D_{\mathcal{E}}^+$ associated to λ. This extends meromorphically to the whole complex plane with the origin not being a pole and $\eta_{\mathcal{E}}^+(0)$ is a well defined real number called the *eta invariant* of $D_{\mathcal{E}}^+$. It measures the overall asymmetry of $\mathrm{Spec}(D_{\mathcal{E}}^+)$ with respect to the origin.

In general, the existence of a boundary implies that the space of harmonic spinors $\ker \partial_{\mathcal{E}} = \ker \partial_{\mathcal{E}}^+ \oplus \ker \partial_{Ec}^-$ is infinite dimensional and one has to impose suitable boundary conditions in order to restore finite dimensionality of kernels. Here we consider Atiyah-Patodi-Singer (APS) type boundary conditions and for this we need to introduce some notation. If $\mathcal{D}$ is a self adjoint elliptic operator acting on sections of a bundle $\mathcal{F} \to \Sigma$, we denote by $\Pi_I(\mathcal{D}) : L^2(\mathcal{F}) \to L^2(\mathcal{F})$ the spectral projection of $\mathcal{D}$ associated to the interval $I \subset \mathbb{R}$. Also, if $\psi \in \Gamma(S_W \otimes \mathcal{E})$ we set $\tilde{\psi} = \psi|_\Sigma$. Now we consider $\Gamma_{\geq 0}(S_W^+ \otimes \mathcal{E}) = \{\psi \in \Gamma(S_W^+ \otimes \mathcal{E}); \Pi_{[0,+\infty)}(D_{\mathcal{E}})\tilde{\psi} = 0\}$ and $\Gamma_{>0}(S_W^- \otimes \mathcal{E}) = \{\psi \in \Gamma(S_W^- \otimes \mathcal{E}); \Pi_{(0,+\infty)}(D_{\mathcal{E}})\tilde{\psi} = 0\}$, which are the domains of the operators

$$\partial\!\!\!/_{\mathcal{E},\geq 0}^+ = \partial\!\!\!/_{\mathcal{E}}^+|_{\Gamma_{\geq 0}(S_W^+ \otimes \mathcal{E})} : \Gamma_{\geq 0}(S_W^+ \otimes \mathcal{E}) \to \Gamma(S_W^- \otimes \mathcal{E}) \tag{2.14}$$

and

$$\partial\!\!\!/_{\mathcal{E},>0}^- = \partial\!\!\!/_{\mathcal{E}}^-|_{\Gamma_{>0}(S_W^- \otimes \mathcal{E})} : \Gamma_{>0}(S_W^- \otimes \mathcal{E}) \to \Gamma(S_W^+ \otimes \mathcal{E}), \tag{2.15}$$

respectively. These are adjoints to each other and moreover $\partial\!\!\!/_{\mathcal{E},\geq 0}^+$ is a Fredholm operator with a well defined index

$$\mathrm{ind}\, \partial\!\!\!/_{\mathcal{E},\geq 0}^+ = \dim\ker \partial\!\!\!/_{\mathcal{E},\geq 0}^+ - \dim\ker \partial\!\!\!/_{\mathcal{E},>0}^-. \tag{2.16}$$

The following formula computes this index (see [APS] or [BoW] for the case where $\mathcal{U}$ is an isometric product and [G] for the general case):

$$\mathrm{ind}\, \partial\!\!\!/_{\mathcal{E},\geq 0}^+ = \int_W [\hat{A}(TW) \wedge \mathrm{ch}(\mathcal{E})]_{2k} + \int_\Sigma [\mathcal{T}\hat{A}(TW) \wedge \mathrm{ch}(\mathcal{E})]_{2k-1} - \xi_{\mathcal{E}}^+(0), \tag{2.17}$$

where $\mathcal{T}\hat{A}$ is the transgression of $\hat{A}$ along Σ, which is polynomial in the curvature, and

$$\xi_{\mathcal{E}}^{+}(0) = \frac{1}{2}\left(\eta_{\mathcal{E}}^{+}(0) + \dim\ker D_{\mathcal{E}}^{+}\right).$$

We now observe that harmonic spinors satisfy the unique continuation property with respect to the pair (W, Σ) and hence $\psi \in \ker \partial_{\mathcal{E}}^{+}$ is completely determined by $\tilde{\psi} \in H^{+}(\Sigma)$, the trace space of such spinors over Σ. Moreover, there exists a continuous bijection $K_{+} : H^{+}(\Sigma) \to \ker \partial_{\mathcal{E}}^{+}$, the so-called Poisson map, which recovers the harmonic spinor given its boundary values. Similar remarks hold for negative spinors and the conclusion is that for $b < 0$ (and with the obvious notation),

$$\operatorname{ind} \partial\!\!\!/_{\mathcal{E},\geq 0}^{+} = \dim\ker \partial\!\!\!/_{\mathcal{E},\geq b}^{+} - \dim\ker \partial\!\!\!/_{\mathcal{E},\geq b}^{-} - \mathcal{I}_{b}, \tag{2.18}$$

where

$$\mathcal{I}_{b} = \dim\operatorname{rank}\Pi_{[b,0)}(D_{\mathcal{E}}^{+}) + \dim\operatorname{rank}\Pi_{(b,0]}(D_{\mathcal{E}}^{-}). \tag{2.19}$$

We now observe that if $\lambda \in [b, 0] \cap \operatorname{Spec}(D_{\mathcal{E}}^{\pm})$ then $\lambda^{2} \in [0, b^{2}] \cap \operatorname{Spec}(D_{\mathcal{E}}^{\pm 2})$ and the proposition below follows immediately.

Proposition 2.1. *In the notation above,* $\mathcal{I}_{b} \leq \dim\operatorname{rank}\Pi_{[0,b^{2}]}(D_{\mathcal{E}}^{2})$.

In order to get rid of the other two terms in the righthand side of (2.18) we now discuss the consequences of a twisted version of an integral identity derived in [HMZ] in the context of the theory described above. We continue assuming that W is an even dimensional compact spin manifold with nonempty boundary Σ and take $\psi \in \Gamma(S_{W} \otimes \mathcal{E})$. Recalling that $\tilde{\psi} = \psi|_{\Sigma}$ the identity reads

$$\int_{W}\left(\frac{\kappa}{4}|\psi|^{2} + \langle \mathcal{R}^{[\mathcal{E}]}\psi, \psi\rangle + |\nabla\psi|^{2} - |\partial\!\!\!/_{\mathcal{E}}\psi|^{2}\right) = \int_{\Sigma}\left(\langle D_{\mathcal{E}}\tilde{\psi}, \tilde{\psi}\rangle - \frac{H}{2}|\tilde{\psi}|^{2}\right).$$

The proof is the same as in [HMZ] and uses (2.2) instead of (2.11). From (2.1) we find

$$|\nabla\psi|^{2} \geq \frac{1}{2k}|\partial\!\!\!/_{\mathcal{E}}\psi|^{2},$$

so that

$$\int_{W}\left(\frac{\kappa}{4}|\psi|^{2} + \langle \mathcal{R}^{[\mathcal{E}]}\psi, \psi\rangle - \frac{2k-1}{2k}|\partial\!\!\!/_{\mathcal{E}}\psi|^{2}\right) \leq \int_{\Sigma}\left(\langle D_{\mathcal{E}}\tilde{\psi}, \tilde{\psi}\rangle - \frac{H}{2}|\tilde{\psi}|^{2}\right).$$

Now expand

$$\tilde{\psi} = \sum_{j} a_{j}\varphi_{j}, \tag{2.20}$$

where $\{\varphi_{j}\}$ is an orthonormal basis for $L^{2}(\mathbb{S}_{\mathcal{E}})$ of eigenspinors of $D_{\mathcal{E}}$: $D_{\mathcal{E}}\varphi_{j} = \lambda_{j}\varphi_{j}$. If $\Pi_{\geq b}\tilde{\psi} = 0$ then $\lambda_{j} < b$, and we obtain the following vanishing result for harmonic spinors satisfying suitable APS boundary conditions.

Proposition 2.2. *Using the notation above, assume that $\kappa/4 + \mathcal{R}^{[\mathcal{E}]} \geq c$ for some $c > 0$, $\psi \in \ker \not{\partial}^{\pm}_{\mathcal{E},\geq b}$ and $a \leq \inf_{\Sigma} H/2$. Then*

$$c \int_W |\psi|^2 \leq (b-a) \int_{\Sigma} |\tilde{\psi}|^2. \tag{2.21}$$

In particular, $\psi \equiv 0$ if $b < a$.

This follows once again from the unique continuation property which gives $\psi \equiv 0$ if and only if $\tilde{\psi} \equiv 0$.

3 Gromov's K-area

The concept of K-area was introduced by Gromov [G1] in order to quantify previous results on geometric-topological obstructions to the existence of metrics with positive curvature. We now briefly review this material.

Let (X, g) be a closed Riemannian manifold (not necessarily spin) of even dimension. Basic results in K-theory [K] show that the set of complex vector bundles over X which are homologically non-trivial (i.e. which have at least a nonzero Chern number) is nonempty. Notice that by Chern-Weil theory [KN] the Chern numbers, which are topological invariants of $\mathcal{E}$, can be computed by integrating over X certain universal differential forms depending on the curvature tensor $R^{\mathcal{E}}$ of any compatible connection on $\mathcal{E}$. Thus $\mathcal{E}$ is homologically trivial (i.e. all Chern numbers vanish) if $R^{\mathcal{E}} = 0$. We then let $\mathcal{E}$ vary over the set of homologically non-trivial hermitian bundles (and compatible connections) over X and define the K-area of (X, g) by

$$K_{\mathrm{area}}(X, g) = \sup \frac{1}{\|R^{\mathcal{E}}\|}, \tag{3.22}$$

where

$$\|R^{\mathcal{E}}\| = \sup_{v\wedge w\neq 0} \frac{\|R^{\mathcal{E}}_{v,w}\|}{\|v\wedge w\|_g} \tag{3.23}$$

and $\|v \wedge w\|_g^2 = g(v, v)g(w, w) - g(v, w)^2$. By the remarks above one always has $K_{\mathrm{area}}(X, g) > 0$. Clearly, the K-area as defined above is a Riemannian invariant but the fact that it is finite or infinite is a homotopy invariant property of X. Simple examples of manifolds with infinite K-area are tori and surfaces of higher genus, whereas simply connected manifolds always have finite K-area.

If X is open (not necessarily complete) one retains the definition (3.22) but restricts to bundles which are trivial at infinity (i.e. in a neighborhood of the point at infinity in the one-point compactification of X). The allowable connections are required to be flat at infinity so that characteristic numbers related to $\mathcal{E}$ are obtained

by integrating over X characteristic differential forms with compact support. Bundles meeting these conditions will be called *admissible*.

Remark 3.1. *Notice however that in the open case a bundle may be trivial without being homologically trivial: this happens when it extends as a homologically non-trivial bundle to the one-point compactification of the base space. This is the case for example of the restriction to $\mathbb{R}^2$ of the Hopf bundle over the Riemann sphere.*

In any case, with this definition, the fact that the K-area is finite or infinite is obviously a *proper* homotopy type invariant of X. Moreover, the following useful characterizations are readily derived from the definitions.

Proposition 3.1. *i) $K_{\rm area}(X) = +\infty$ if and only if for any $\epsilon > 0$ there exists a homologically non-trivial admissible $\mathcal{E}$ over X with $\|R^{\mathcal{E}}\| \leq \epsilon$; ii) $K_{\rm area}(X, g) < +\infty$ if and only if there exists $\epsilon_{X,g} > 0$ such that if $\mathcal{E}$ over X is admissible and $\|R^{\mathcal{E}}\| \leq \epsilon_{X,g}$ then $\mathcal{E}$ is homologically trivial.*

Remark 3.2. *Since the one-point compactification of $\mathbb{R}^{2k}$ is the unit sphere S^{2k} and $K_{\rm area}(S^{2k}) < +\infty$ then $K_{\rm area}(\mathbb{R}^{2k}) < +\infty$ as well. Now recall that a Riemannian manifold X^{2k} is hyper-euclidian if there exists a proper Lipschitz map $f : X \to \mathbb{R}^{2k}$ with nonzero degree. This promptly gives $K_{\rm area}(X) \geq {\rm Lip}(f)^{-2} K_{\rm area}(\mathbb{R}^{2k})$ and sending ${\rm Lip}(f) \to 0$ (which can be accomplished by scaling) we get $K_{\rm area}(X) = +\infty$. Thus if we further assume $X \in \mathcal{BG}_{2k}$, Proposition 1.1 applies to X.*

4 Uniformly finite homology and bounded de Rham cohomology

As remarked in the Introduction, Block and Weinberger [BW] have defined a bounded geometry (in fact, coarse) invariant homology H_0^{uf}, the so-called *uniformly finite homology* in degree zero. If $Y \in \mathcal{BG}_n$ then $S \subset Y$ defines a class $[S] \in H_0^{uf}(Y)$ if S is locally uniformly finite (see the Introduction).

In what follows, a crucial property of H_0^{uf} is that it is naturally dual to H_b^n, the *bounded de Rham cohomology* in degree $n = \dim Y$ [W]. If $\xi \in H_b^n(Y)$ we denote by $\xi^{uf}(Y)$ the corresponding class in $H_0^{uf}(Y)$. A closer look at the proof of the Poincaré duality $H_0^{uf} \cong H_b^n$ in [W] reveals that it behaves quite well under the infinite connected sum operations we are dealing with.

Proposition 4.1. *If $\xi \in H_b^n(Y)$ is given by a characteristic form via Chern-Weil theory, $[S] \in H_0^{uf}(Y)$ and Z is closed then*

$$\xi^{uf}(Y \sharp_S Z) = \xi^{uf}(Y) + \xi(Z)[S], \tag{4.24}$$

where $\xi(Z) = \int_Z \xi$ is the characteristic number computed over Z.

Here, (4.24) should of course be interpreted in the sense of Remark 1.1.

Remark 4.1. *It is also observed in [BW] that $H_0^{uf}(Y)$ is trivial if and only Y is open at infinity, which means by definition that domains in Y satisfy a linear isoperimetric inequality. Since this certainly is not the case for the manifold $Y = V^{2k-2} \times P^2$ in Remark 1.3, it follows that Theorem 1.1 applies for example to a quasi-lattice $S \subset Y$.*

5 The proofs of Propositions 1.1 and 1.2

We start by giving the proof of Proposition 1.1. We argue by contradiction and assume the existence of $X \in \mathcal{BG}_{2k}$ with $K_{\mathrm{area}}(X) = +\infty$ and carrying a metric g with $\kappa > 0$. We proceed by borrowing a trick from [GL] and considering the (not necessarily complete) metric $g_\kappa = \kappa g$. Hence, for any $\epsilon > 0$, there exists an admissible *homologically nontrivial* bundle $\mathcal{E}$ over X with $\|R^{\mathcal{E}}\|_\kappa \leq \epsilon$. The notation $\| \ \|_\kappa$ of course means that the norm (3.23) is computed with respect to g_κ, so if g is used instead we get $\|R^{\mathcal{E}}\| \leq \epsilon\kappa$ pointwisely. In view of (2.3) this gives $\|\mathcal{R}^{[\mathcal{E}]}\| \leq \epsilon\rho_{2k}\kappa$ for some $\rho_{2k} > 0$ depending only on the dimension. It follows that

$$\mathcal{R}^{[\mathcal{E}]} + \frac{\kappa}{4} \geq \left(-\epsilon\rho_{2k} + \frac{1}{4}\right)\kappa,$$

so if $\epsilon \leq 1/8\rho_{2k}$ we obtain the pointwise estimate

$$\mathcal{R}^{[\mathcal{E}]} + \frac{\kappa}{4} \geq \frac{\kappa}{8}. \tag{5.25}$$

We now remark that integration over the fundamental cycle of X defines a class $[\hat{A}(TX) \smile \mathrm{ch}(\mathcal{E})]_{2k} \in H_b^{2k}(X)$. We are going to show that this class vanishes (or equivalently that $[\hat{A}(TX) \wedge \mathrm{ch}(\mathcal{E})]_{2k} = d\omega$ for ω a uniformly bounded $(2k-1)$-form) under the given conditions. By results in [W] this boils down to show that

$$\left| \int_W [\hat{A}(TX) \wedge \mathrm{ch}(\mathcal{E})]_{2k} \right| \leq C\mathrm{vol}_{2k-1}(\Sigma), \tag{5.26}$$

for $W \subset X$ a compact regular domain with $\Sigma = \partial W$, with the constant C depending only on given bounds for the second fundamental form of Σ and its covariant derivatives.

The first step in checking (5.26) is to use (5.25) and Proposition 2.2 with $c = \inf_W \kappa/8 > 0$ and $b = 2a$ (here we may assume $a < 0$). It follows that both

$\ker \not{\partial}^+_{\mathcal{E},\geq 2a}$ and $\ker \not{\partial}^-_{\mathcal{E},\geq 2a}$ are trivial so $\operatorname{ind} \not{\partial}^+_{\mathcal{E},\geq 2a} = -\mathcal{I}_{2a}$ by (2.18). We now use this in conjunction with the index formulae (2.17) to estimate

$$\left|\int_W [\hat{A}(TX)\wedge \mathrm{ch}(\mathcal{E})]_{2k}\right| \leq \left|\int_\Sigma [\mathcal{T}\hat{A}(TX)\wedge \mathrm{ch}(\mathcal{E})]_{2k-1}\right| + + \left|\xi^+_{\mathcal{E}}(0)\right| + \mathcal{I}_{2a}. \tag{5.27}$$

Now the curvature bounds easily imply

$$\left|\int_\Sigma [\mathcal{T}\hat{A}(TX)\wedge \mathrm{ch}(\mathcal{E})]_{2k-1}\right| \leq C\mathrm{vol}_{2k-1}(\Sigma),$$

and it is known that $|\xi^+_{\mathcal{E}}(0)| \leq C\mathrm{vol}_{2k-1}(\Sigma)$ as well [R]. As for $\mathcal{I}_{2a}$ we know from Proposition 2.1 that it is bounded by the sum of multiplicities of the eigenvalues of the Dirac Laplacian $D^2_{\mathcal{E}}$ in the interval $[0, 4a^2]$ and, given the curvature bounds at our disposal, this can be estimated from above by $C_a\mathrm{vol}_{2k-1}(\Sigma)$ just as in [W], so that in view of (5.27), (5.26) follows immediately. Thus, $[\hat{A}(TX)\smile \mathrm{ch}(\mathcal{E})]_{2k} = 0$ in $H^{2k}_b(X)$ and we are in a position to evoke Gromov's algebraic argument [G1] based on Adam's operations to deduce that $\mathcal{E}$ is homologically trivial. This contradiction proves Proposition 1.1.

As for Proposition 1.2, set $L = Y\sharp_S Z$ and assume by contradiction that $K_{\mathrm{area}}(L) < +\infty$. If g is a metric in the given bounded geometry class, by Proposition 3.1 there exists $\epsilon_{L,g} > 0$ such that if $\mathcal{E}$ admissible over L satisfies $\|R^{\mathcal{E}}\| \leq \epsilon_{L,g}$ then $\mathcal{E}$ is homologically trivial. On the other hand, since $K_{\mathrm{area}}(Z) = +\infty$, for any $0 < \epsilon < \epsilon_{L,g}$ there exists a homologically nontrivial $\mathcal{E}'$ over Z with $\|R^{\mathcal{E}'}\| \leq \epsilon$. Now, if $\mathcal{V} \subset Z$ is a compact tubular neighborhood of the sphere S^{2k-1} over which the connected sum operation leading to L was carried out, then $\mathcal{V}$ has the same homotopy type as S^{2k-1} and by Bott periodicity there exists p such that $\mathcal{E}'' = \mathcal{E}' \oplus \Theta_p$ is trival over $\mathcal{V}$ (here, $\Theta_p = Z \times \mathbb{C}^p$, which is endowed with the standard flat connection). Thus $\mathcal{E}''$ can be extended both to Y and L as trivial, and hence admissible, bundles. Since $\|R^{\mathcal{E}''}\| \leq \epsilon < \epsilon_{L,g}$, $\mathcal{E}''$ is homologically trivial over L by the contradiction assumption. In particular, by Poincaré duality, $c_I(\mathcal{E}'')^{uf}(L) = 0$ for *any* Chern number class c_I. On the other hand, since $\mathcal{E}'$ and $\mathcal{E}''$ have the same Chern classes, $\mathcal{E}''$ is homologically nontrivial over Z. This allows us to choose c_I such that $c_I(\mathcal{E}'')(Z) \neq 0$ and use Proposition 4.1 to get $c_I(\mathcal{E}'')^{uf}(Y) = -c_I(\mathcal{E}'')(Z)[S] \neq 0$, so that $\mathcal{E}''$ is *not* homologically trivial over Y. But this means that $K_{\mathrm{area}}(Y) = +\infty$ and this contradiction proves Proposition 1.2.

Remark 5.1. *Very likely, Proposition 1.1 (and hence Theorem 1.1) holds with 'positive' replaced by 'nonnegative', but our method fails to deal with this case. A positive result in this direction would cover for example manifolds like $\mathbb{R}^{2k}\sharp_{\mathbb{Z}^{2k}}\mathbb{T}^{2k}$, where $\mathbb{Z}^{2k} \subset \mathbb{R}^{2k}$ is the standard lattice.*

Remark 5.2. *The methods above also apply to closed manifolds, so we get the analogue of Theorem 4.2 in [W]: if Z^{2k} is a closed spin manifold of infinite K-area then, given bounds on the curvature, any metric on Z meeting these bounds can not have the nonpositive part of its scalar curvature contained in an arbitrarily small neighborhood of a point.*

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